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Published in:
Mathematische zeitschrift

Publication date:
1978

Math. Z. 160, 217–234 (1978)

Mathematische Zeitschrift

Polarizations in the Classical Groups

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1. Introduction

1.1. Let G be a connected reductive group over an algebraically closed field k. It has an adjoint action on its Lie algebra $\mathfrak{g}$. Let P be a parabolic subgroup of G. Let $U(P)$ denote its unipotent radical with Lie algebra $\mathfrak{u}(P)$. An element $x \in \mathfrak{u}(P)$ is called a *Richardson element* of $\mathfrak{u}(P)$ if $\dim(Gx) = 2\dim(G/P)$.

Lemma. *Let* $x \in \mathfrak{u}(P)$.
(a) $\dim(Gx) \leqq 2\dim(G/P)$.
(b) *x is a Richardson element of $\mathfrak{u}(P)$ if and only if Px is dense in $\mathfrak{u}(P)$ and P contains the identity component $Z_G(x)^0$ of the centralizer $Z_G(x)$.*

Proof. This follows from the inequality

$$\begin{aligned}\dim(Gx) &= \dim(G) - \dim(Z_G(x)) \leqq \dim(G) - \dim(Z_P(x)) = \\ &= \dim(G/P) + \dim(Px) \leqq \dim(G/P) + \dim(\mathfrak{u}(P)) = 2\dim(G/P).\end{aligned}$$

1.2. Assume that $\mathfrak{g}$ has only finitely many orbits of nilpotent elements. It is known that this assumption is fulfilled if G is a classical group or if the characteristic of k is good enough.

Theorem (a) (Richardson [8, 11] (3.9)). *Every parabolic group P in G has Richardson elements in $\mathfrak{u}(P)$.*
(b) (Borho-Jantzen [3] (5.18)). *Let P_1 and P_2 be parabolic groups in G. For $i = 1, 2$ let $x_i \in \mathfrak{u}(P_i)$ be a Richardson element and let L_i be a Levi factor of P_i. If L_1 is conjugate to L_2 in G then x_1 is conjugate to x_2 under G.*

1.3. Conversely, let $x \in \mathfrak{g}$ be nilpotent. A parabolic group P in G is called a *polarization* of x if x is a Richardson element of $\mathfrak{u}(P)$. The set of polarizations of x is denoted by $\mathrm{Pol}(x)$.

Remark. So we only consider polarizations of nilpotent elements. A more general definition leads to questions which are usually first reduced to the nilpotent case, cf. the proof of [3] (5.16). The structure of $\mathrm{Pol}(x)$ is interesting for the infinite-dimensional representation theory, cf. [7] and [1] (6.3).

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By (1.1) (b) the "disconnected" centralizer $A(x) := Z_G(x)/Z_G(x)^0$ acts on the set $\mathrm{Pol}(x)$. If $P \in \mathrm{Pol}(x)$ its stabilizer $A_P(x)$ in $A(x)$ is equal to $Z_P(x)/Z_G(x)^0$, since parabolic subgroups are self normalizing, cf. [6] (23.1).

Lemma. *Let $\mathscr{P}$ be a conjugacy class of parabolic subgroups of G. If $\mathscr{P} \cap \mathrm{Pol}(x)$ is non-empty then it is one single orbit under the finite group $A(x)$.*

Proof. Let $P, P' \in \mathscr{P} \cap \mathrm{Pol}(x)$, say $P' = \mathrm{int}(g)P$. Both x and $g^{-1}x$ are Richardson elements of $\mathfrak{u}(P)$. Their dense P-orbits in $\mathfrak{u}(P)$ intersect, so there is $p \in P$ with $px = g^{-1}x$. We have $gp \in Z_G(x)$ and $P' = \mathrm{int}(gp)P$.

Corollary. $\mathrm{Pol}(x)$ *is a finite set.*

Proof. G has only finitely many conjugacy classes of parabolic subgroups.

1.4. We shall describe $\mathrm{Pol}(x)$ for the classical groups: $G = \mathrm{Gl}(n)$, or $\mathrm{char}(k) \neq 2$ and $G = \mathrm{SO}(n)$ or $G = \mathrm{Sp}(n)$. We verify Theorem (1.2) by hand. So it is natural to describe $\mathrm{Pol}(x)$ in the following steps.

Step 1. Describe the conjugacy classes $\mathscr{L}$ of the Levi factors of the elements of $\mathrm{Pol}(x)$.

Step 2. Given class $\mathscr{L}$ of step 1, determine the classes $\mathscr{P}$ of parabolic subgroups with Levi factor in $\mathscr{L}$. The number of classes $\mathscr{P}$ associated to $\mathscr{L}$ is denoted by $N_0 = N_0(\mathscr{L})$.

Step 3. Given a class $\mathscr{P}$ of step 2, describe $\mathscr{P} \cap \mathrm{Pol}(x)$ which set is an orbit under $A(x)$, by a representative $P \in \mathscr{P} \cap \mathrm{Pol}(x)$ together with its stabilizer $A_P(x)$ in $A(x)$. The number of elements of $\mathscr{P} \cap \mathrm{Pol}(x)$ is denoted by $N_1 = N_1(\mathscr{P})$. So N_1 is the index of $A_P(x)$ in $A(x)$, or equivalently of $Z_P(x)$ in $Z_G(x)$.

1.5. For the classical groups the finite group $A(x)$ is always abelian. So the stabilizer $A_P(x)$ is independent of the choice of $P \in \mathscr{P} \cap \mathrm{Pol}(x)$. It turns out that $A_P(x)$ only depends on the class $\mathscr{L}$. More precisely we obtain.

Corollary (7.6). *For $i = 1, 2$ let $P(i) \in \mathrm{Pol}(x)$, say with Levi factor $L(i)$. The following conditions are equivalent:*

(a) *$L(1)$ is conjugate to $L(2)$ in G.*

(b) *$A_{P(1)}(x) = A_{P(2)}(x)$ (or equivalently $Z_{P(1)}(x) = Z_{P(2)}(x)$).*

(c) *$N_1(P(1)) = N_1(P(2))$.*

Note that (b)$\Rightarrow$(c) is trivial. The implication (a)$\Rightarrow$(c) holds in general, cf. [4] (7.2). These results lead to the following conjecture for a reductive group G and a nilpotent element $x \in \mathfrak{g}$.

Conjecture. *Let $P(1)$, $P(2) \in \mathrm{Pol}(x)$ have Levi factors $L(1)$ and $L(2)$. Then $L(1)$ is conjugate to $L(2)$ in G if and only if $Z_{P(1)}(x)$ is conjugate to $Z_{P(2)}(x)$ in $Z_G(x)$.*

1.6. The solution of the steps 1 and 2 is due to N. Spaltenstein, cf. [9] Section 5. His methods are sufficient to solve step 3 as well. We give a new proof and more explicit results. Finally we give tables for the groups of type B_2, B_3, B_4, C_2, C_3, C_4, D_4, D_5, D_6.

It is a pleasure to express our gratitude to Walter Borho and Hanspeter Kraft for a stimulating correspondence, many discussions and much hospitality.

2. Combinatorial Conventions

$\mathbb{N}$ is the set of natural numbers $1, 2, 3, \ldots$. If V is a finite set, its cardinality is denoted by $\#(V)$. If x is a real number $[x]$ denotes the largest integer $\leqq x$.

We define a *partition* λ to be a finite subset of $\mathbb{N}^2$ such that if $(p, q) \in \lambda$ and $i \leqq p$ and $j \leqq q$ then $(i, j) \in \lambda$. So we identify a partition with the corresponding Young diagram. If λ is a partition we define.

$$|\lambda| = \#(\lambda), \quad \lambda_j = \#\{i \mid (i,j) \in \lambda\}, \quad \lambda^i = \#\{j \mid (i,j) \in \lambda\}.$$

A partition λ is determined by each of the non-increasing sequences λ_* and λ^*. Clearly $|\lambda| = \sum_j \lambda_j = \sum_i \lambda^i$. One verifies that

$$\sum_j (j-1)\lambda_j = \sum_i \tfrac{1}{2}\lambda^i(\lambda^i - 1). \tag{2.1}$$

If $(p_1, \ldots, p_s)$ is a finite sequence in $\mathbb{N}$ we define the partition $\lambda = \operatorname{ord}(p_1, \ldots, p_s)$ by $\lambda_j = \#\{i \mid p_i \geqq j\}$. Then $\lambda_1 = s$ so that $\lambda^s > 0 = \lambda^{s+1}$. There is a permutation σ of $\{1, \ldots, s\}$ such that $\lambda^i = p_{\sigma(i)}$ for $1 \leqq i \leqq s$. In particular $|\lambda| = \sum p_i$.

For a partition λ we define $N(\lambda)$ to be the number of sequences $(p_1, \ldots, p_s)$ with $\lambda = \operatorname{ord}(p_1, \ldots, p_s)$. Since the above permutation σ is unique up to multiplication by elements of the stabilizer of $(\lambda^1, \ldots, \lambda^s)$ we have

$$N(\lambda) = \lambda_1! \prod_{i \geqq 1} ((\lambda_i - \lambda_{i+1})!)^{-1}. \tag{2.2}$$

Let G be a reductive group. We define $b(G) = \dim(G/B)$ where B is some (any) Borel group of G. If P is a parabolic subgroup of G with Levi factor L then we have

$$\dim(G/P) = b(G) - b(L). \tag{2.3}$$

For reference below we note the formulas

$$\begin{aligned} &b(\mathrm{Gl}(n)) = \tfrac{1}{2}n(n-1) \\ &b(\mathrm{Sp}(n)) = \tfrac{1}{2}(\tfrac{1}{2}n(n-1) + [\tfrac{1}{2}n]) \quad \text{(here } n \text{ is even)} \\ &b(\mathrm{SO}(n)) = \tfrac{1}{2}(\tfrac{1}{2}n(n-1) - [\tfrac{1}{2}n]). \end{aligned} \tag{2.4}$$

3. The General Linear Group Gl(n)

3.1. Let V be a vector space over the algebraically closed field k with $\dim(V) = n$. Let G be $\mathrm{Gl}(V)$. Its Lie algebra $\mathfrak{g}$ is identified with $\operatorname{End}(V)$. Let $x \in \mathfrak{g}$ be nilpotent. The endomorphism x has a Jordan normal form. So there is a partition λ with $|\lambda| = n$ and a *Jordan basis* of V consisting of vectors $e(i, j)$, $(i, j) \in \lambda$, such that $x\,e(i, j) = e(i-1, j)$ if $i > 1$, and $x\,e(1, j) = 0$. If $m \geqq 0$ then $\dim(\operatorname{Im}(x^m)) = \sum_{i>m} \lambda^i$. So the partition λ of x is unique. G acts on $\mathfrak{g}$ by $\operatorname{Ad}(g)\,x = g x g^{-1}$. It is clear that nilpotent elements x and y of $\mathfrak{g}$ are conjugate under G if and only if they have the same partition. The centralizer $Z_G(x)$ of an element $x \in \mathfrak{g}$ is open in the linear space $\{y \in \operatorname{End}(V) | yx = xy\}$, and hence connected.

3.2. A (partial) *flag* F in V is a sequence of subspaces $(F_0, \ldots, F_s)$ with $F_0 = 0$, $F_s = V$ and $F_{i-1} \subset F_i$, $F_{i-1} \neq F_i$ for $1 \leqq i \leqq s$. Its *type* is the sequence $(p_1, \ldots, p_s)$ given by $p_i = \dim(F_i/F_{i-1})$. Its *stabilizer* P consists of the $g \in G$ with $gF_i = F_i$ for $0 \leqq i \leqq s$. Now P is a parabolic subgroup of G. Every parabolic subgroup P of G is the stabilizer of a unique flag. The type of its flag is called the *flag type* of P. Parabolic subgroup of G are conjugate if and only if they have the same flag type.

Let P be the stabilizer of the flag $F = (F_0, \ldots, F_s)$. Its unipotent radical consists of the elements $g \in P$ with $(g-1)\, F_i \subset F_{i-1}$ for $1 \leqq i \leqq s$. So $\mathfrak{u}(P)$ consists of the $x \in \operatorname{End}(V)$ with $x F_i \subset F_{i-1}$ for $1 \leqq i \leqq s$. If we choose factors L_i with $F_i = F_{i-1} \oplus L_i$ then

$$L = \prod_{i=1}^{s} \operatorname{Gl}(L_i) \cong \prod_{i=1}^{s} \operatorname{Gl}(p_i)$$

is a Levi factor of P. So parabolic subgroups P and Q of G with flag types $(p_1, \ldots, p_s)$ and $(q_1, \ldots, q_t)$ have conjugate Levi factors if and only if $\operatorname{ord}(p_1, \ldots, p_s) = \operatorname{ord}(q_1, \ldots, q_t)$, cf. Section 2.

3.3. Theorem. *Let $x \in \mathfrak{g}$ be nilpotent with partition λ.*

(a) *The element x has a polarization P with flag type $(p_1, \ldots, p_s)$ if and only if $\lambda = \operatorname{ord}(p_1, \ldots, p_s)$. In that case P contains $Z_G(x)$ and is unique.*

(b) *The element x has $N(\lambda)$ polarizations. All its polarizations have conjugate Levi factors, isomorphic to $\prod_{i \geqq 1} \operatorname{Gl}(\lambda^i)$.*

Proof. Although this is known, cf. [9] (5.3), we indicate a proof as a preparation for the other classical groups.

Assume $\lambda = \operatorname{ord}(p_1, \ldots, p_s)$. We construct a flag $F = (F_0, \ldots, F_s)$ with flag type $(p_1, \ldots, p_s)$ such that $x F_i \subset F_{i-1}$ for $1 \leqq i \leqq s$. Put $p = p_1$ and $m = \lambda_p - 1$ and $F_1 = \operatorname{Ker}(x) \cap \operatorname{Im}(x^m)$. If $e(i,j)$, $(i,j) \in \lambda$, is a Jordan basis of V with respect to x, cf. (3.1), then $F_1 = \sum_{j=1}^{p} k e(1,j)$. It follows that $\dim(F_1) = p_1$. The induced endomorphism x' of V/F_1 has partition μ given by $\mu_j = \lambda_j - 1$ if $j \leqq p$, and $\mu_j = \lambda_j$ if $j > p$. One verifies that $\mu = \operatorname{ord}(p'_1, \ldots, p'_{s-1})$ where $p'_i = p_{i+1}$. By induction we may assume the existence of a flag $F' = (F'_0, \ldots, F'_{s-1})$ in V/F_1 of type $(p'_1, \ldots, p'_{s-1})$ such that $x' F'_i \subset F'_{i-1}$ for $1 \leqq i \leqq s-1$. Writing $F'_i = F_{i+1}/F_1$ we obtain a flag F which satisfies the requirements. Let P be the stabilizer of F. Then we have $x \in \mathfrak{u}(P)$. Using the formulas (2.3), (2.4) and (2.1) we obtain

$$2 \dim(G/P) = n^2 - n - 2 \sum (j-1)\, \lambda_j.$$

It follows that $2 \dim(G/P) = \dim(Gx)$, as calculated for example in [5] (3.8). So P is a polarization. Since $Z_G(x)$ is connected P contains $Z_G(x)$. Alternatively this may be verified by the induction.

By the conjugacy of the parabolic subgroup of a given flag type, it follows that every parabolic subgroup, say with flag type $(q_1, \ldots, q_t)$, is the above constructed polarization of some nilpotent element with partition $\operatorname{ord}(q_1, \ldots, q_t)$. By the P-conjugacy of the Richardson elements of $\mathfrak{u}(P)$ this proves part (a). Part (b) follows immediately. See Section 2 for the definition of $N(\lambda)$.

Remark. In the proof we had to verify (1.2) (a) by hand. The assertion (1.2) (b) follows also.

4. The Symplectic and the (Special) Orthogonal Group

We treat these groups in a unified way, cf. [10] p. 253 and [5] (3.1).

4.1. Conventions. All congruences are modulo 2. We fix a number ε equal to 0 or 1. The field k is algebraically closed and of characteristic $\neq 2$. Let V be a vector space over k with $\dim(V)=n$. Let $\varphi\colon V \times V \to k$ be a non-degenerate bilinear form satisfying

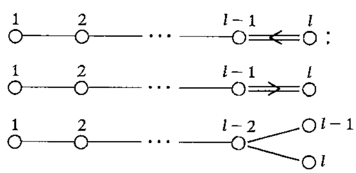
$$\varphi(v_1, v_2) = (-1)^{\varepsilon} \varphi(v_2, v_1) \qquad v_1, v_2 \in V.$$

Remark. (a) So $n \equiv 0$ if $\varepsilon = 1$. (b) All forms φ are equivalent.

The group $H = H(V, \varphi)$ consists of the elements $g \in \mathrm{Gl}(V)$ which preserve the form φ. The group $G = G(V, \varphi)$ is the identity component of H. It consists of the elements of H with determinant 1. The common Lie algebra $\mathfrak{g}$ of G and H consists of the elements $x \in \operatorname{End}(V)$ satisfying

$$\varphi(x v_1, v_2) + \varphi(v_1, x v_2) = 0 \qquad v_1, v_2 \in V.$$

The groups G and H are reductive of rank $l = [\frac{1}{2} n]$. Their types, Dynkin diagrams and classical names are as follows.



$n \not\equiv \varepsilon = 1$ $\quad C_l$: $\quad G = H = \mathrm{Sp}(n)$

$n \not\equiv \varepsilon = 0$ $\quad B_l$

$n \equiv \varepsilon = 0$ $\quad D_l$ $\quad G = \mathrm{SO}(n) \neq H = \mathrm{O}(n)$.

4.2. We can choose a basis $e_1, \ldots, e_n$ of V such that $\varphi(e_i, e_j) \neq 0$ if and only if $i+j = n+1$. The torus T in G with weight vectors $e_1, \ldots, e_n$ is maximal. Let $\lambda_1, \ldots, \lambda_n$ be the corresponding weights. They generate the character group $X(T)$ and are subject to the relations $\lambda_i + \lambda_{n+1-i} = 0$, and $\lambda_{l+1} = 0$ if $n = 2l+1$.

The root system $\Phi = \Phi(G, T)$ consists of the differences $\lambda_i - \lambda_j$ with $i \neq j$, and $i+j \neq n+1$ if $\varepsilon = 0$. The simple roots $\alpha_1, \ldots, \alpha_l$ are chosen $\alpha_i = \lambda_i - \lambda_{i+1}$ if $1 \leq i < l$, $\alpha_l = \lambda_l - \lambda_{l+1} = (1+\varepsilon)\lambda_l$ if $n \not\equiv \varepsilon$, and $\alpha_l = \lambda_{l-1} - \lambda_{l+1} = \lambda_{l-1} + \lambda_l$ if $n \equiv \varepsilon = 0$. Their numbers correspond to the numbers in the above Dynkin diagram.

Remark. (a). Assume $n \equiv \varepsilon = 0$. Let $r \in H$ be the reflexion given by $r e_i = e_i$ if $i \notin \{l, l+1\}$, and $r e_l = e_{l+1}$ and $r e_{l+1} = e_l$. This reflexion interchanges the roots α_{l-1}

and α_l. The group H is generated by r and G. (b) If $n \not\equiv \varepsilon = 0$ then H is generated by G and $-\mathrm{id}$, which element usually acts trivial.

4.3. A flag $F=(F_0, \ldots, F_s)$ in V is called *isotropic* if $F_i^\perp = F_{s-i}$ for $0 \leqq i \leqq s$. Let $(p_1, \ldots, p_s)$ be its type, cf. (3.2). It satisfies $p_i = p_{s+1-i}$ for $1 \leqq i \leqq s$. Put $t=[\frac{1}{2}s]$, and $q=0$ if $s=2t$, and $q=p_{t+1}$ if $s=2t+1$. Clearly $q \equiv n$. One verifies the existence of a basis $e_1, \ldots, e_n$ of V such that

$$\varphi(e_i, e_j) \neq 0 \Leftrightarrow i+j=n+1,$$

$$F_j = \sum k e_i, \qquad 1 \leqq i \leqq \dim(F_j).$$

We assume that this is the basis of (4.2). Consider the full isotropic flag $F' = (F'_0, \ldots, F'_n)$ given by $F'_j = \sum_{i=1}^{j} k e_i$. Let P and B be the stabilizers in G of the flags F and F', respectively, cf. (3.2). The group B is the Borel group of G corresponding to the basis $\alpha_1, \ldots, \alpha_l$ of Φ. So P is some standard parabolic group P_I, cf. [6] (30.1).

We say that the flag F is *not admissible* if there is a P-invariant subspace F' and an index j with $F_j \subset F' \subset F_{j+1}$ and $F_j \neq F' \neq F_{j+1}$.

Lemma. (a) *The flag F is admissible if and only if $\varepsilon = 1$ or $q \neq 2$.*

(b) *Assume that F is admissible. Then $P = P_I$ where I consists of the indices i with $1 \leqq i \leqq l$ and $i \neq \dim(F_j)$ for all j.*

(c) *Every parabolic subgroup of G is the stabilizer of some admissible isotropic flag.*

Proof. (a) and (b) are left to the reader. (c) Every parabolic subgroup is conjugate to some P_I with $I \subset \{1, \ldots, l\}$. By Remark (4.2) (a) we may assume that $n \not\equiv \varepsilon$ or that $l-1 \in I$ or that $l \notin I$. By (a) all such sets I occur in (b).

4.4. By Lemma (4.3)(b) the type $(p_1, \ldots, p_s)$ of an admissible isotropic flag with stabilizer P is a well defined invariant of P. It is called the *flag type* of P.

Lemma. *Parabolic subgroups in G are conjugate under the action of H if and only if they have the same flag type. The class of parabolic subgroups of G with flag type $(p_1, \ldots, p_s)$ splits into two conjugacy classes under the action of G if and only if $\varepsilon = 0$ and $s = 2t$ and $p_t \geqq 2$.*

4.5. Let $F=(F_0, \ldots, F_s)$ be an admissible isotropic flag of type $(p_1, \ldots, p_s)$. Let P be its stabilizer in G. Let L be a Levi factor of P and let S be the radical of L. So S is a maximal torus of the radical of P and L is the centralizer of S in G, cf. [6] (30.2). Consider the weight space decomposition $V = \sum V(\chi)$ of V with respect to S. For $\omega \in X(S)$ we have

$$V(\omega)^\perp = \sum_{\chi \neq -\omega} V(\chi).$$

The group L consists of the $g \in G$ with $g\,V(\chi)=V(\chi)$ for all $\chi \in X(S)$. One verifies that $X(S)$ has a basis $\chi_1, \dots, \chi_t$ where $t=[\frac{1}{2}s]$ such that

$$V=V(0)\oplus\sum_{i=1}^{t}(V(\chi_i)\oplus V(-\chi_i)),$$

$$F_j=\sum_{i=1}^{j}V(\chi_i) \quad \text{if } j\leqq t.$$

We may identify

$$L=G(V(0),\varphi\,|\,V(0))\times\prod_{i=1}^{t}\mathrm{Gl}(V(\chi_i)).$$

In the notations of (4.3) we have

$$p_i=\dim(V(\chi_i)) \quad \text{if } i\leqq t,$$
$$q=\dim(V(0)).$$

We put $\nu=\mathrm{ord}(p_1, \dots, p_t)$, cf. Section 2. The pair $(\nu; q)$ is called the *type* of L, or the *Levi type* of P.

Remark. If $\varepsilon=0$ then $\dim(V(0))\neq 2$, since $\mathrm{SO}(2)$ is a non-trivial torus, see Lemma (4.3)(a).

4.6. An integer q is called *admissible* if $q\geqq 0$, and $q\equiv n$, and $q\neq 2$ if $\varepsilon=0$. We define $\mathrm{Lev}(n)$ to be the set of pairs $(\nu; q)$ such that q is admissible and ν is a partition with $n=q+2|\nu|$.

Lemma. (a) $\mathrm{Lev}(n)$ *is the set of the Levi types of the parabolic subgroups of* G.

(b) *Levi factors of parabolic subgroups of* G *are conjugate under the action of* H *if and only if they have the same type.*

(c) *The class of Levi factors of a given type* $(\nu; q)$ *splits into two conjugacy classes under the action of* G *if and only if* $q=\varepsilon=0$ *and* $\nu^i\equiv 0$ *for all* i.

(d) *The number of conjugacy classes* $\mathscr{P}$ *of parabolic subgroups of* G *with given Levi type* $(\nu; q)$ *is equal to* $N(\nu)$ *if* $q+\varepsilon\geqq 1$, *and to* $(1+\nu_1^{-1}\nu_2)\,N(\nu)$ *if* $q=\varepsilon=0$.

Proof. (a) and (b) are immediate. (c) Choose a Levi factor L of type $(\nu; q)$. The class splits if and only if $H\neq G$ and the normalizer N of L in H is contained in G. The first condition is equivalent to $\varepsilon=0$. The normalizer N is generated by L and $H(V(0),\varphi\,|\,V(0))$ and some standard mappings interchanging weight spaces $V(\pm\chi_i)$, $1\leqq i\leqq t$. We have $H(V(0),\varphi\,|\,V(0))\subset G$ if and only if $q=0$. All interchanging mappings have determinant 1 if and only if $\nu^i\equiv 0$ for all i.

(d) $N(\nu)$ is the number of types $(p_1, \dots, p_s)$ of isotropic flags in V such that $\nu=\mathrm{ord}(p_1, \dots, p_t)$ where $t=[\frac{1}{2}s]$, cf. Section 2. By Lemma (4.4) this settles the case $q+\varepsilon\geqq 1$. Assume $q=\varepsilon=0$. By (4.4) the required number is equal to $2N(\nu)-N'$ where N' is the number of sequences $(p_1, \dots, p_t)$ with $p_t=1$ and $\nu=\mathrm{ord}(p_1, \dots, p_t)$. If $\nu_1=\nu_2$ then $N'=0$. If $\nu_1>\nu_2$ then $N'=N(\tau)$ where τ is the partition given by $\tau_1=\nu_1-1$ and $\tau_j=\nu_j$ for $j\geqq 2$. By (2.2) we have

$$N(\tau)=\nu_1^{-1}(\nu_1-\nu_2)\,N(\nu).$$

5. Nilpotent Elements

5.1. Let $x \in \mathfrak{g}$ be nilpotent. By (3.1) there is a unique partition λ and a *Jordan basis* $e(i,j)$, $(i,j) \in \lambda$, such that $x e(i,j) = e(i-1,j)$ if $i > 1$ and $x e(1,j) = 0$. By [10] page 259, this basis can be *normalized* such that we have $\varphi(e(i,j), e(p,q)) \neq 0$ if and only if $i + p = \lambda_j + 1$ and $q = \beta(j)$, where β is some permutation of $\{1, \dots, \lambda^1\}$. By the symmetry relations of (4.1), if follows that $\beta^2 = \mathrm{id}$ and $\lambda_{\beta(j)} = \lambda_j$, and that $\beta(j) \neq j$ whenever $\lambda_j \equiv \varepsilon$. Using Remark (4.1)(b) one verifies that any permutation β satisfying these conditions can be used for x.

Let $\mathrm{Pan}(n, \varepsilon)$ denote the set of the partitions λ such that $|\lambda| = n$ and that the number $\# \{j \mid \lambda_j = m\}$ is even for every $m \in \mathbb{N}$ with $m \equiv \varepsilon$. Again using Remark (4.1)(b) we obtain

Lemma. (a) *A partition λ is the partition of some nilpotent element $x \in \mathfrak{g}$ if and only if $\lambda \in \mathrm{Pan}(n, \varepsilon)$.*

(b) *Nilpotent elements x and y of $\mathfrak{g}$ are conjugate under H if and only if they have the same partition.*

5.2. Let $x \in \mathfrak{g}$ be nilpotent with partition λ and normalized Jordan basis as in (5.1). We put $M_i = \mathrm{Ker}(x) \cap \mathrm{Im}(x^{i-1})$. The vectors $e(1,j)$ with $1 \leqq j \leqq \lambda^i$ form a basis of M_i, so that $\dim(M_i) = \lambda^i$. One verifies the existence of a unique bilinear form φ_i on M_i such that $\varphi_i(v, v') = \varphi(v, v'')$ if $v, v' \in M_i$ and $v' = x^{i-1} v''$. This form satisfies

$$\varphi_i(v_1, v_2) = (-1)^{i-1+\varepsilon} \varphi_i(v_2, v_1) \qquad \text{if } v_1, v_2 \in M_i.$$

One verifies that M_{i+1} is the kernel of the form φ_i on M_i. So there is an induced non-degenerate bilinear form φ_i' on M_i/M_{i+1}. Let $C_i = H(M_i/M_{i+1}, \varphi_i')$ be the corresponding orthogonal or symplectic group, cf. (4.1). The canonical morphism

$$Z_H(x) \to \prod_i C_i$$

is surjective. Its kernel is connected. If $\lambda^i = \lambda^{i+1}$ or $i \equiv \varepsilon$, the group C_i is connected. If $\lambda^i > \lambda^{i+1}$ and $i \not\equiv \varepsilon$ then C_i is an orthogonal group with two connected components. Compare [10] page 261.

5.3. Consider the finite groups $A(x) = Z_G(x)/Z_G(x)^0$ and $A'(x) = Z_H(x)/Z_H(x)^0$, cf. (1.3). We have $A(x) \subset A'(x)$. By (5.2) the group $A'(x)$ is abelian and all its elements have order 2. We may consider $A'(x)$ as a vector space over $\mathbb{F}_2$ with a basis indexed by the pairs (i, λ^i) such that $\lambda^i > \lambda^{i+1}$ and $i \not\equiv \varepsilon$. This index set is equal to the set of pairs (λ_j, j) such that

$$j \in B(\lambda) = \{j \in \mathbb{N} \mid \lambda_j > \lambda_{j+1}, \lambda_j \not\equiv \varepsilon\}.$$

So we may identify

$$A'(x) = \{a \in \mathbb{F}_2^{\mathbb{N}} \mid j \notin B(\lambda) \Rightarrow a_j = 0\}.$$

If $\varepsilon=1$ then $G=H$ so that $A(x)=A'(x)$. If $\varepsilon=0$ the group G consists of the elements $g\in H$ with determinant 1. In general we obtain

$$A(x)=\{a\in \mathbb{F}_2^{\mathbb{N}}\,|\,j\notin B(\lambda)\Rightarrow a_j=0;\ \varepsilon=0\Rightarrow \textstyle\sum a_j=0\}.$$

Remark. Assume, as we may, that the normalization of (5.1) satisfies $\beta(j)=j$ if and only if $\lambda_j\neq\varepsilon$. For $m\in\mathbb{N}$ with $\lambda_m\neq\varepsilon$ we define the reflexion $r(m)\in Z_H(x)$ by

$$r(m)\,e(i,j)=e(i,j) \qquad \text{if } m\neq j,$$
$$r(m)\,e(i,j)=-e(i,j) \qquad \text{if } m=j.$$

The image of $r(m)$ in $A'(x)$ is the vector $r'(m)$ with $r'(m)_j=1$ if and only if $\lambda_m=\lambda_j>\lambda_{j+1}$. So the vectors $r'(j)$ with $j\in B(\lambda)$ form a basis of $A'(x)$. Compare [9] (3.2), (3.9), (3.14).

5.4. Corollary. *Let $x\in\mathfrak{g}$ be nilpotent with partition λ. The G-orbit of x is unequal to the H-orbit of x if and only if $\varepsilon=0$ and $B(\lambda)$ is empty. In that case the H-orbit of x consists of two G-orbits.*

Proof. This is the only case where $G\neq H$ and $A(x)=A'(x)$. Compare [10] page 264.

6. Combinatorics

6.1. Let $\mathrm{Pai}(n,q)$ denote the set of partitions π such that $\pi_j\equiv 1$ if $j\leqq q$, and $\pi_j\equiv 0$ if $j>q$. We define the set $\mathrm{Pai}(n)$ to be the (clearly disjoint) union of the sets $\mathrm{Pai}(n,q)$ with q admissible, cf. (4.6). We identify the set $\mathrm{Lev}(n)$ of (4.6) with $\mathrm{Pai}(n)$ using the bijection $(v;q)\mapsto\pi$ given by

$$\pi_j=2v_j+1 \quad \text{if } j\leqq q, \qquad \pi_j=2v_j \quad \text{if } j>q. \tag{$*$}$$

So we write $(v;q)=\pi$ if $(*)$ holds.

Remark. If P is a parabolic group with flag type $(p_1,\dots,p_s)$, cf. (4.4), we have thus identified its Levi type $(v;q)$ with the partition $\pi=\mathrm{ord}(p_1,\dots,p_s)$.

6.2. The set $\mathrm{Pan}(n,\varepsilon)$ is defined in (5.1). We define the Spaltenstein mapping $S\colon \mathrm{Pai}(n)\to\mathrm{Pan}(n,\varepsilon)$ as follows. If $\pi\in\mathrm{Pai}(n)$, put

$$I(\pi)=\{j\in\mathbb{N}\,|\,j\neq n,\ \pi_j\equiv\varepsilon,\ \pi_j\geqq\pi_{j+1}+2\}$$

and let the partition $\lambda=S(\pi)$ be given by

$$\lambda_j=\pi_j-1 \qquad \text{if } j\in I(\pi),$$
$$\lambda_j=\pi_j+1 \qquad \text{if } j-1\in I(\pi),$$
$$\lambda_j=\pi_j \qquad \text{otherwise}.$$

It is clear that λ is a partition with $|\lambda|=n$. In (6.4) we verify that $\lambda\in\mathrm{Pan}(n,\varepsilon)$.

6.3. Lemma. *Let $\pi \in \operatorname{Pai}(n, q)$, q admissible, and $\lambda = S(\pi)$.*

(a) *Assume $\lambda_j \neq \pi_j$. Then $\lambda_j = \pi_j + (-1)^{n-j}$. We have $j \leqq q$ if $\varepsilon = 1$, and $j > q$ if $\varepsilon = 0$. In either case $\pi_j \equiv \varepsilon \not\equiv \lambda_j$.*

(b) $\# \{j \mid \lambda_j \equiv 1\} = q + 2(-1)^{\varepsilon} \# I(\pi)$.

(c) *If $j \not\equiv n$ and $\lambda_j > \lambda_{j+1}$ then $\lambda_j \equiv \lambda_{j+1} \not\equiv \varepsilon$; if $n \equiv 1$ then $\lambda_1 \not\equiv \varepsilon$.*

Proof. (a) Since $q \equiv n$ we have $q \notin I(\pi)$. (b) follows from (a).

(c) Let $j \not\equiv n$ and $\lambda_j > \lambda_{j+1}$. It follows from (a) that $\pi_j > \pi_{j+1}$. Since $j \neq q$ we have $\pi_j \equiv \pi_{j+1}$, so that $\pi_j \geqq \pi_{j+1} + 2$. If $\pi_j \equiv \varepsilon$ then $j \in I(\pi)$ and the assertion follows. If $\pi_j \not\equiv \varepsilon$ then $j \notin I(\pi)$ and again the assertion follows. Assume $n \equiv 1$. Then $\varepsilon = 0$ by Remark (4.1)(a), and $q \neq 0$. It follows that $\pi_1 \equiv 1$ and that $\lambda_1 = \pi_1$.

6.4. For a partition λ with $|\lambda| = n$ we consider the condition $C(\lambda)$: if $j \not\equiv n$ then $\lambda_j \equiv \lambda_{j+1}$.

Lemma. *The assertion* (6.3)(c) *holds if and only if $C(\lambda)$ holds and $\lambda \in \operatorname{Pan}(n, \varepsilon)$.*

The proof may be left to the reader.

6.5. For a partition λ with $|\lambda| = n$ we define

$$J(\lambda) = \{j \mid \lambda_j \equiv \varepsilon\} \cup \{j, j+1 \mid j \equiv n, \lambda_j = \lambda_{j+1}\},$$
$$j_1(\lambda) = \sup \{j \in J(\lambda) \mid \lambda_j \equiv 1\} \quad (\text{possibly } -\infty),$$
$$j_0(\lambda) = \min \{j \in J(\lambda) \mid \lambda_j \equiv 0\}.$$

Proposition. *Let q be an admissible integer and $\lambda \in \operatorname{Pan}(n, \varepsilon)$.*

(a) *The restriction S: $\operatorname{Pai}(n, q) \to \operatorname{Pan}(n, \varepsilon)$ is injective.*

(b) *We have $\lambda \in S(\operatorname{Pai}(n, q))$ if and only if $C(\lambda)$ holds and $j_1(\lambda) \leqq q < j_0(\lambda)$.*

(c) *We have $\lambda \in S(\operatorname{Pai}(n))$ if and only if $C(\lambda)$ holds and $j_1(\lambda) < j_0(\lambda)$.*

Proof. (a) follows from (6.3)(a).

(b) Assume $\lambda = S(\pi)$ with $\pi \in \operatorname{Pai}(n, q)$. Condition $C(\lambda)$ holds by (6.3) (c) and (6.4). If $j \equiv n$ and $\lambda_j = \lambda_{j+1}$, then $\lambda_j \geqq \pi_j \geqq \pi_{j+1} \geqq \lambda_{j+1}$ so that $\lambda_j = \pi_j$ and $\lambda_{j+1} = \pi_{j+1}$. Together with (6.3)(a) this proves that $\lambda_j = \pi_j$ whenever $j \in J(\lambda)$. This implies $j_1(\lambda) \leqq q < j_0(\lambda)$.

Assume that $C(\lambda)$ holds and that $j_1(\lambda) \leqq q < j_0(\lambda)$. Let π_* be the unique sequence in $\mathbb{Z}$ with $\pi_j = \lambda_j - (-1)^{n-j} \delta_j$ where $\delta_j \in \{0, 1\}$ such that $\pi_j \equiv 1$ if and only if $j \leqq q$. The inequality implies that $\pi_j = \lambda_j$ whenever $j \in J(\lambda)$.

If $\lambda_j \geqq \lambda_{j+1} + 2$ then $\pi_j \geqq \pi_{j+1}$. If $\lambda_j = \lambda_{j+1} + 1$, then $j \in J(\lambda)$ or $j+1 \in J(\lambda)$, so that $\pi_j \geqq \pi_{j+1}$. If $j \equiv n$ and $\lambda_j = \lambda_{j+1}$, then $j \in J(\lambda)$ and $j+1 \in J(\lambda)$, so that $\pi_j = \pi_{j+1}$. If $j \not\equiv n$ then $\pi_j \geqq \lambda_j \geqq \lambda_{j+1} \geqq \pi_{j+1}$. So the sequence π_* is non-increasing. If $\lambda_j = 0$ then $\pi_i = 0$ for all $i > j$. This proves that π_* defines a partition π.

If $j \not\equiv n$ then $\lambda_j \equiv \lambda_{j+1}$ and $\pi_j \equiv \pi_{j+1}$, by $C(\lambda)$ and $q \equiv n$. It follows that $\pi_j = \lambda_j + 1$ if and only if $\pi_{j+1} = \lambda_{j+1} - 1$. Using that $n \equiv 1$ implies $\varepsilon = 0$, one proves that $\pi_1 \neq \lambda_1 - 1$. It follows that $|\pi| = |\lambda| = n$, so that $\pi \in \operatorname{Pai}(n, q)$.

Moreover it now suffices to prove that $j \in I(\pi)$ if and only if $\pi_j = \lambda_j + 1$. By the last paragraph, $\pi_j = \lambda_j + 1$ implies $j \in I(\pi)$. Assume $j \in I(\pi)$ and $\pi_j \neq \lambda_j + 1$. We have $j \not\equiv n$ and $\pi_j \equiv \varepsilon$ and $\pi_j \geqq \pi_{j+1} + 2$ and $\pi_j = \lambda_j$ and $\pi_{j+1} = \lambda_{j+1}$. Condition (6.3)(c) holds by (6.4), so that $\pi_j = \lambda_j \not\equiv \varepsilon$, a contradiction.

(c) Assume that $C(\lambda)$ holds and that $j_1(\lambda)<j_0(\lambda)$. It remains to show the existence of an admissible integer q with $j_1(\lambda)\leqq q<j_0(\lambda)$. If $\varepsilon=1$ then $n\equiv 0$ and either $j_1(\lambda)=-\infty$ or $j_1(\lambda)\equiv n$ by (6.3)(c), so that we may use $q=0$ or $q=j_1(\lambda)$ respectively. Assume $\varepsilon=0$. By (6.3)(c) we have $j_0(\lambda)\not\equiv n$. If $j_0(\lambda)\neq 3$ we may use $q=j_0(\lambda)-1$. If $j_0(\lambda)=3$ then $n\equiv 0$ and $j_1(\lambda)=-\infty$ so that we may use $q=0$.

6.6. Lemma. *Assume* $\varepsilon=0$. *Let* $(v;q)=\pi\in\mathrm{Pai}(n)$ *and* $\lambda=S(\pi)$.

(a) *The set* $B(\lambda)$ *of* (5.3) *is empty if and only if* $q=0$ *and* $\lambda=\pi$, *if and only if* $q=0$ *and* $v^i\equiv 0$ *for all* i.

(b) *We have* $q>0$ *if and only if there is* $j\in B(\lambda)$ *such that* $j\notin I(\pi)$ *and* $j-1\notin I(\pi)$.

The proof is left to the reader.

7. The Main Theorem

7.1. Let $x\in\mathfrak{g}$ be nilpotent with partition λ, cf. (5.1). We use the description of its disconnected centralizer $A(x)$ given in (5.3). Let $\pi=(v;q)$ be an element of $\mathrm{Pai}(n)$, cf. (6.1). Let $\mathrm{Pol}(x,\pi)$ denote the set of polarizations P of x with Levi type $(v;q)$. The Spaltenstein mapping S is defined in (6.2) and analysed in (6.5). If $\lambda=S(\pi)$, we put

$$u=\#I(\pi)=\tfrac{1}{2}(-1)^{\varepsilon}(-q+\#\{j\mid\lambda_j\equiv 1\}),\qquad \text{cf. (6.3)(b).}$$

Theorem. (a) $\mathrm{Pol}(x,\pi)$ *is non-empty if and only if* $\lambda=S(\pi)$.

(b) *If* $P\in\mathrm{Pol}(x,\pi)$ *its stabilizer* $A_P(x)$ *in* $A(x)$ *consists of the elements* $a\in A(x)$ *with* $a_j=a_{j+1}$ *whenever* $j\in I(\pi)$.

(c) *Levi factors of elements of* $\mathrm{Pol}(x,\pi)$ *are conjugate in* G.

(d) *If* $\lambda=S(\pi)$ *then* $\mathrm{Pol}(x,\pi)$ *consists of* N_0 *conjugacy classes under* $A(x)$ *each consisting of* N_1 *polarizations where*

$$N_0=N(v)\quad\text{and}\quad N_1=2^u\quad\text{if } q+\varepsilon\geqq 1\quad\text{or}\quad B(\lambda)\quad\text{is empty.}$$

$$N_0=(1+v_1^{-1}v_2)N(v)\quad\text{and}\quad N_1=2^{u-1}\quad\text{if } q=\varepsilon=0\quad\text{and}\quad B(\lambda)\neq\varnothing.$$

7.2. Lemma. *Let* P *be a parabolic subgroup with Levi type* $(v;q)=\pi$. *Assume that* $\lambda=S(\pi)$. *Then* $\dim(Gx)=2\dim(G/P)$.

Proof. Let L be a Levi factor of P. By (2.3), (2.4) and (4.6) we have $2\dim(G/P)=2b(G)-2b(L)$ where

$$\begin{aligned}2b(L)&=\sum v^i(v^i-1)+\tfrac{1}{2}q(q-1)-(-1)^{\varepsilon}[\tfrac{1}{2}q]\\&=\sum\tfrac{1}{2}\pi^i(\pi^i-1)-(-1)^{\varepsilon}[\tfrac{1}{2}q].\end{aligned}$$

By [5] (3.8) we have $\dim(Gx)=2b(G)-\gamma_{\varepsilon}(\lambda)$ where

$$\gamma_{\varepsilon}(\lambda)=\sum(j-1)\lambda_j-(-1)^{\varepsilon}[\tfrac{1}{2}\#\{j\mid\lambda_j\equiv 1\}].$$

Now the result follows from (2.1) and (6.3)(b).

7.3. Lemma. *Assume that $\lambda=S(\pi)$. Let $(p_1,\dots,p_s)$ be a sequence in $\mathbb{N}$ with $\pi=\operatorname{ord}(p_1,\dots,p_s)$ and $p_i=p_{s+1-i}$ for $1\leqq i\leqq s$. Then x has a polarization P with flag type $(p_1,\dots,p_s)$.*

Proof. It suffices to construct an isotropic flag $F=(F_0,\dots,F_s)$ of type $(p_1,\dots,p_s)$ such that $xF_i\subset F_{i-1}$ for $1\leqq i\leqq s$. In fact by the assumption $\pi\in\operatorname{Pai}(n)$ the flag F will be admissible. So its stabilizer P in G is parabolic with flag type $(p_1,\dots,p_s)$. The condition $xF_i\subset F_{i-1}$ implies that $x\in\mathfrak{u}(P)$, so P is a polarization of x by (7.2).

Assume $s=1$. Then $\pi_j\leqq 1$ for all j. So $I(\pi)$ is empty and $\lambda=\pi$ and $x=0$. We choose $F_0=0$, $F_1=V$.

Assume $s>1$. Put $p=p_1=p_s$. The type $(p_2,\dots,p_{s-1})$ of the required isotropic flag $(F_i/F_1)_{0<i<s}$ in F_{s-1}/F_1 gives rise to a partition $\rho\in\operatorname{Pai}(n-2p,q)$ given by

$$\rho_j=\#\{i\mid 2\leqq i\leqq s-1, p_i\geqq j\},$$

so that $\rho_j=\pi_j-2$ if $j\leqq p$, and $\rho_j=\pi_j$ if $j>p$.

Let S: $\operatorname{Pai}(n-2p)\to\operatorname{Pan}(n-2p,\varepsilon)$ be the Spaltenstein mapping, cf. (6.2). Put $\mu=S(\rho)$. There are two cases.

(A) $I(\pi)=I(\rho)$. Then $p\equiv n$ or $\pi_p\not\equiv\varepsilon$ or $\pi_p\geqq\pi_{p+1}+4$. We have $\mu_j=\lambda_j-2$ if $j\leqq p$, and $\mu_j=\lambda_j$ if $j>p$. Note that $\lambda_p>\lambda_{p+1}$ since μ is a partition.

(B) $I(\pi)=\{p\}\cup I(\rho)$ and $p\notin I(\rho)$. Then $p\not\equiv n$, $\pi_p\equiv\varepsilon$ and $\pi_p=\pi_{p+1}+2$. We have $\mu_j=\lambda_j-2$ if $j<p$, and $\mu_j=\lambda_j$ if $j>p+1$, and $\mu_p=\lambda_p-1$, $\mu_{p+1}=\lambda_{p+1-1}$. Moreover $\lambda_p=\lambda_{p+1}\not\equiv\varepsilon$ and $\lambda_{p+1}>\lambda_{p+2}$.

We choose a normalized Jordan basis $e(i,j)$, $(i,j)\in\lambda$, of V with respect to x, cf. (5.1). In case (B) we require moreover that $\beta(p)=p+1$. We put $F_0=0$, $F_s=V$. In case (A) we define $F_1=\sum_{j=1}^{p} k e(1,j)$. In case (B) we have two possibilities

$$F_1=\sum_{j=1}^{p} k e(1,j) \quad\text{or}\quad F_1=\sum_{j=1,j\neq p}^{p+1} k e(1,j)$$

In both cases we put $F_{s-1}=F_1^{\perp}$.

One verifies that $xF_1=0$ and hence $xF_s\subset F_{s-1}$, that F_1 is contained in F_{s-1}, and that the induced endomorphism x_1 of F_{s-1}/F_1 has partition μ. We proceed by induction.

7.4. Lemma. *Let P be the polarization of x constructed in (7.3). The stabilizer $A_P(x)$ consists of the elements $a\in A(x)$ with $a_j=a_{j+1}$ whenever $j\in I(\pi)$.*

Proof. Let F be the flag constructed in (7.3). The group $Z_H(x)^0$ is contained in P and therefore it leaves F invariant. So we may define the stabilizer $A'_F(x)$ of F in $A'(x)$. We have $A_P(x)=A(x)\cap A'_F(x)$.

It suffices to prove that $a\in A'(x)$ leaves F invariant if and only if $a_j=a_{j+1}$ whenever $j\in I(\pi)$. The case $s=1$ is trivial since $I(\pi)$ is empty and $A'(x)=A'_F(x)$. We proceed by induction as in (7.3). Clearly any element of $A'(x)$ leaves F_0 and F_s invariant.

Assume $s>1$. Put $p=p_1=p_s$ and $h=\lambda_p$. In case (A) we have $F_1=M_h$, cf. (5.2). In case (B) the space F_1 is one of the two hyperplanes in M_h with $\varphi_h|F_1=0$. It follows that $a\in A'(x)$ leaves F_1 and F_{s-1} invariant if and only if case (A) applies or $a_{p+1}=0$ and case (B) applies.

Assume that $a\in A'(x)$ leaves F_1 and F_{s-1} invariant. Then a has an image $a'\in A'(x_1)$ where x_1 is the induced endomorphism of F_{s-1}/F_1. The element a leaves F invariant if and only if a' leaves the induced flag $(F_i/F_1)_{0<i<s}$ in F_{s-1}/F_1 invariant. By induction this is the case if and only if $a'_j=a'_{j+1}$ whenever $j\in I(\rho)$.

Case (A_1). $I(\pi)=I(\rho)$ and $B(\lambda)=B(\mu)$, so that $\lambda_p\equiv\varepsilon$ or $\lambda_p>\lambda_{p+1}+2$. After the identifications of (5.3) the map $A'(x)\to A'(x_1)$ is the identity. One verifies this using the basis of Remark (5.3). Now the assertion follows by induction.

Case (A_2). $I(\pi)=I(\rho)$ and $B(\lambda)=\{p\}\cup B(\mu)$ where $p\notin B(\mu)$. So $\lambda_p\not\equiv\varepsilon$ and $\lambda_p=\lambda_{p+1}+2$. First assume that $\lambda_p>2$. We have a unique $i>p$ such that $\mu_p=\mu_{p+1}=\mu_i>\mu_{i+1}$. Moreover $i\in B(\mu)$. In the notation of Remark (5.3) we have that the images of $r(p)$ and $r(i)$ in $A'(x_1)$ are both equal to $r'(i)$. It follows that $a'\in A'(x_1)$ is given by $a'_j=a_j$ if $j\notin\{p,i\}$ and $a'_p=0$ and $a'_i=a_p+a_i$. Now it suffices to verify that $a_j-a_{j+1}=a'_j-a'_{j+1}$ whenever $j\in I(\rho)$. In fact $I(\pi)=I(\rho)$, so we may proceed by induction. Let $j\in I(\rho)$. Then we have $\mu_j\geqq\mu_{j+1}>\mu_{j+2}$ and $\mu_{j-1}>\mu_j$ if $j>1$. It follows that $j\leqq p-2$, or $j=p=i-1$, or $j>i$. Since the coordinates are in $\mathbb{F}_2$ the required verification is trivial. If $\lambda_p=2$ the proof is similar.

Case (B). $I(\pi)=\{p\}\cup I(\rho)$ and $p\notin I(\rho)$. Then we have $p\notin B(\lambda)$, so $a_p=0$ and the element $a\in A'(x)$ leaves F_1 and F_{s-1} invariant if and only if $a_p=a_{p+1}$. Assume $a_p=a_{p+1}=0$. Again using Remark (5.3) one verifies that $a'=a$. If $j\in I(\pi)$ and $j\neq p$ then $j\leqq p-2$ or $j\geqq p+2$. So the induction is immediate.

7.5. *The proof of* (7.1) *and the verification of* (1.2) *in our case.*

Since H acts transitive on the set of parabolic subgroups of G with a given flag type, cf. Lemma (4.4), Lemma (7.3) implies that every parabolic subgroup of G admits Richardson elements, cf. (1.2)(a). The same argument shows that every polarization of x is obtained by a construction as in (7.3), compare the proof of (1.3). This proves (7.1)(a). Now Lemma (7.4) implies (7.1)(b).

By (4.6)(c), the assertion (7.1)(c) is only then non-trivial if $\lambda=S(\pi)$ and $q=\varepsilon=0$ and $v^i\equiv 0$ for all i. Then $B(\lambda)$ is empty, cf. (6.6), so that the H-orbit of x splits into two G-orbits, cf. (5.4). Let $(p_1,\dots,p_s)$ be as in (7.3). Put $t=\frac{1}{2}s=v_1$. The construction of (7.3) yields that F_t is generated by the vectors $e(i,j)$ with $i\leqq\frac{1}{2}\lambda_j$. So we have

$$F_t=\sum_m x^m\,\mathrm{Ker}(x^{2m})$$

Since this space is independent of the choice of $(p_1,\dots,p_s)$ the result (7.1)(c) follows easily. Now one may verify (1.2)(b).

Assume that $\lambda=S(\pi)$. By (1.2) the number N_0 of (7.1)(d) can be obtained from (4.6) taking into account the splitting of the Levi type, cf. (4.6) and (6.6)(a). The number N_1 is equal to the index of $A_P(x)$ in $A(x)$. So we have $N_1=2^v$ where v is the dimension of the vector space $A(x)/A_P(x)$. If $j\in I(\pi)$ then $j\neq n$ and $j+1\in B(\lambda)$. This implies that

$$\dim(A'(x)/A'_F(x))=\#I(\pi)=u$$

in the notations of (7.4). It follows that

$$v \neq u \Leftrightarrow v = u-1 \Leftrightarrow A'_F(x) \subset A(x) \neq A'(x).$$

We have $A(x)=A'(x)$ if and only if $\varepsilon=1$ or $B(\lambda)$ is empty. So assume that $\varepsilon=0$ and $B(\lambda)$ is non-empty. Using (6.6)(b) one verifies that $A'_F(x)$ is not contained in $A(x)$ if and only if $q>0$.

7.6. Corollary. *For $i=1,2$ let $P(i)$ be a polarization of x, say with Levi factor $L(i)$. Let $N_1(i)$ be the number of polarizations of x conjugate to $P(i)$ under $Z_G(x)$. The following conditions are equivalent.*

(a) *$L(1)$ and $L(2)$ are conjugate in G.*

(b) *$A_{P(1)}(x)=A_{P(2)}(x)$.*

(c) *$N_1(1)=N_1(2)$.*

Proof. (a)$\Rightarrow$(b) by (7.1)(b). The implication (b)$\Rightarrow$(c) is trivial.

Assume that (c) holds. Write $N_1(1)=N_1(2)=2^v$. Let $\pi(i)=(v(i); q(i))$ be the Levi type of $P(i)$, so that $\lambda=S(\pi(1))=S(\pi(2))$. Put $u(i)=\# I(\pi(i))$. By (7.1)(d) we have

$$\begin{aligned} u(i)&=v \qquad &&\text{if } q(i)+\varepsilon \geqq 1 \text{ or } B(\lambda)=\varnothing,\\ u(i)&=v+1 \qquad &&\text{if } q(i)=\varepsilon=0 \text{ and } B(\lambda)\neq\varnothing. \end{aligned}$$

Suppose that $q(1)\neq q(2)$, say $q(1)<q(2)$. By (6.3)(b) we have

$$u(1)-u(2)=\tfrac{1}{2}(-1)^{\varepsilon}(q(2)-q(1))\neq 0.$$

It follows that $\varepsilon=0$ and $q(1)=0$ and $u(1)-u(2)=1$, so that $q(2)=2$, contradicting the fact that $q(2)$ should be admissible, cf. (6.1). This proves that $q(1)=q(2)$. By (6.5)(a) it follows that $\pi(1)=\pi(2)$ so that (a) follows from (7.1)(c).

7.7. A polarization P of x is called *stable* if P contains $Z_G(x)$. This concept seems to be important in the applications, cf. [2] (6.7)(3).

Corollary. *The element x has stable polarizations if and only if either* (i) $\lambda\in\mathrm{Pai}(n)$, *or* (ii) $n\equiv\varepsilon=0$ *and* $\{j\mid\lambda_j\equiv 1\}=\{r,r+1\}$ *for some* $r\equiv 1$. *Assume* (i) *or* (ii) *holds. The stable polarizations of x are the N_0 elements of* $\mathrm{Pol}(x,\pi)$ *where in case* (i) *we have $\pi=\lambda$, and in case* (ii) *we have $\pi_j=\lambda_j$ if $j\notin\{r,r+1\}$ and $\pi_r=\lambda_r+1$ and $\pi_{r+1}=\lambda_{r+1}-1$.*

The proof is left to the reader. Note that the conditions (i) and (ii) are mutually exclusive.

Remark. If x is an even nilpotent element, cf. [10] page 241, the parabolic subgroup P considered in loc.cit. is a stable polarization of x. The Levi type of P and hence of all stable polarizations of x, can be deduced from the weighted Dynkin diagram of x, cf. loc.cit. page 243, 263 and 264.

7.8. Corollary. *Assume $n\not\equiv\varepsilon$ (i.e. G is of type B_l or C_l).*

(a) *The polarizations of x have Levi factors with the same semisimple rank.*

(b) *If P_1 and P_2 are parabolic groups such that their Levi factors have the same Dynkin diagram, then their Richardson elements are conjugate.*

The proof is left to the reader. The table for D_5 in (7.9) clearly shows that both assertions are false for type D_l.

7.9. Tables. A conjugacy class of Levi factors is characterized by the Dynkin type in terms of A_l, B_l, C_l, D_l. The symbols B_1, C_1, D_2, D_3 are used to distinguish certain parts of the Dynkin diagram of G, though formally $B_1 = C_1 = A_1$, $D_2 = A_1 + A_1$, $D_3 = A_3$. The same information is carried by the Levi type $(v; q)$. The partition v is given by the sequence $v^* = (v^1, \dots, v^t)$ where $v^t > 0 = v^{t+1}$.

The numbers N_0 and N_1 are meant conform (1.4) and (7.1). The partition $\lambda = S(\pi)$ of a Richardson element x is given by the sequence $\lambda_* = (\lambda_1, \dots, \lambda_m)$ where $\lambda_m > 0 = \lambda_{m+1}$. The groups $A(x)$ and $A_P(x)$ are determined by the sets $B(\lambda)$ and $I(\pi)$, cf. (5.3) and (7.1). These sets are given in the tables.

We give the weighted Dynkin diagram of x, in view of remark (7.7). If the H-orbit of x or equivalently the Levi type, splits into two conjugacy classes under the action of G, we write $2 \times N_0$ in the column of N_0 (so that this column adds up to 2^l). In this case we give the Dynkin diagram of one of the G-orbits. We indicate how to get the other diagram by the symbol $\curvearrowright$. A partition λ such that the corresponding nilpotent elements have no polarizations at all, is equipped with a dot and dash line in the column of (v^*, q).

Tabelle 1

$B_2 = SO(5)$

Levi	$(v^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*	○⇒○
0	(1 1; 1)	1	1	∅	1	5	2 2
B_1	(1; 3)	1	1	∅	1 3	3 1 1	2 0
A_1	(2; 1)	1	2	2			
	—·—·—				3	2 2 1	0 1
B_2	(–; 5)	1	1	∅	5	1^5	0 0

$C_2 = Sp(4)$

Levi	$(v^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*	○⇐○
0	(1 1; 0)	1	1	∅	1	4	2 2
A_1	(2; 0)	1	1	∅	2	2 2	0 2
C_1	(1; 2)	1	2	1			
	—·—·—				1	2 1 1	1 0
C_2	(–; 4)	1	1	∅	∅	1^4	0 0

$B_3 = SO(7)$

Levi	$(v^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*	○—○⇒○
0	(1 1 1; 1)	1	1	∅	1	7	2 2 2
B_1	(1 1; 3)	1	1	∅	1 3	5 1 1	2 2 0
A_1	(2 1; 1)	2	2	2			
A_1+B_1	(2; 3)	1	1	∅	2 3	3 3 1	0 2 0
A_2	(3; 1)	1	1	∅	1	3 2 2	1 0 1
B_2	(1; 5)	1	1	∅	1 5	3 1^4	2 0 0
	—·—·—				5	2 2 1^3	0 1 0
B_3	(–; 7)	1	1	∅	7	1^7	0 0 0

Table 1 (continued)

$C_3 = \mathrm{Sp}(6)$

Levi	$(\nu^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*	o—o⇐o
0	(1 1 1; 0)	1	1	$\varnothing$	1	6	2 2 2
A_1	(2 1; 0)	2	1	$\varnothing$	1 2	4 2	2 0 2
C_1	(1 1; 2)	1	2	1			
	—·—·—				1	4 1 1	2 1 0
A_1+C_1	(2; 2)	1	1	$\varnothing$	$\varnothing$	3 3	0 2 0
A_2	(3; 0)	1	1	$\varnothing$	3	2 2 2	0 0 2
C_2	(1; 4)	1	2	1	2	2 2 1 1	0 1 0
	—·—·—				1	$2\ 1^4$	1 0 0
C_3	(−; 6)	1	1	$\varnothing$	6	1^6	0 0 0

$B_4 = \mathrm{SO}(9)$

Levi	$(\nu^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*	o—o—o⇒o
0	(1 1 1 1; 1)	1	1	$\varnothing$	1	9	2 2 2 2
B_1	(1 1 1; 3)	1	1	$\varnothing$	1 3	7 1 1	2 2 2 0
A_1	(2 1 1; 1)	3	2	2			
A_1+B_1	(2 1; 3)	2	1	$\varnothing$	1 2 3	5 3 1	2 0 2 0
A_1+A_1	(2 2; 1)	1	2	2			
A_2	(3 1; 1)	2	1	$\varnothing$	1	5 2 2	2 1 0 1
B_2	(1 1; 5)	1	1	$\varnothing$	1 5	$5\ 1^4$	2 2 0 0
	—·—·—				3	4 4 1	0 2 0 1
A_2+B_1	(3; 3)	1	1	$\varnothing$	3	3 3 3	0 0 2 0
A_1+B_2	(2; 5)	1	1	$\varnothing$	2 5	$3\ 3\ 1^3$	0 2 0 0
A_3	(4; 1)	1	2	4	1 5	3 2 2 1 1	1 0 1 0
B_3	(1; 7)	1	1	$\varnothing$	1 7	$3\ 1^6$	2 0 0 0
	—·—·—				5	$2^4\ 1$	0 0 0 1
	—·—·—				7	$2\ 2\ 1^5$	0 1 0 0
B_4	(−; 9)	1	1	$\varnothing$	9	1^9	0 0 0 0

$C_4 = \mathrm{Sp}(8)$

Levi	$(\nu^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*	o—o—o⇐o
0	(1 1 1 1; 0)	1	1	$\varnothing$	1	8	2 2 2 2
A_1	(2 1 1; 0)	3	1	$\varnothing$	1 2	6 2	2 2 0 2
C_1	(1 1 1; 2)	1	2	1			
	—·—·—				1	6 1 1	2 2 1 0
A_1+A_1	(2 2; 0)	1	1	$\varnothing$	2	4 4	0 2 0 2
A_1+C_1	(2 1; 2)	2	2	1			
A_2	(3 1; 0)	2	1	$\varnothing$	1 3	4 2 2	2 0 0 2
C_2	(1 1; 4)	1	2	1	1 2	4 2 1 1	2 0 1 0
	—·—·—				1	$4\ 1^4$	2 1 0 0
A_2+C_1	(3; 2)	1	1	$\varnothing$	3	3 3 2	0 1 1 0
A_1+C_2	(2; 4)	1	1	$\varnothing$	$\varnothing$	3 3 1 1	0 2 0 0
A_3	(4; 0)	1	1	$\varnothing$	4	2^4	0 0 0 2
	—·—·—				3	$2^3\ 1\ 1$	0 0 1 0
C_3	(1; 6)	1	2	1	2	$2\ 2\ 1^4$	0 1 0 0
	—·—·—				1	$2\ 1^6$	1 0 0 0
C_4	(−; 8)	1	1	$\varnothing$	$\varnothing$	1^8	0 0 0 0

Table 1 (continued)

$D_4 = \mathrm{SO}(8)$

Levi	$(\nu^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*	
0	(1 1 1 1; 0)	1	1	1	1 2	7 1	2 2 2 2
A_1	(2 1 1; 0)	4	1	1	1 2	5 3	2 0 2 2
A_1+A_1	(2 2; 0)	2×1	1	$\varnothing$	$\varnothing$	4 4	0 2 2 0
D_2	(1 1; 4)	1	1	$\varnothing$	1 4	5 1 1 1	2 2 0 0
A_1+D_2	(2; 4)	1	1	$\varnothing$ $\}$	2 4	3 3 1 1	0 2 0 0
A_2	(3 1; 0)	3	2	1 3 $\}$			
	—·—·—				1 4	3 2 2 1	1 0 1 1
A_3	(4; 0)	2×1	1	$\varnothing$	$\varnothing$	2 2 2 2	0 0 2 0
D_3	(1; 6)	1	1	$\varnothing$	1 6	3 1^5	2 0 0 0
	—·—·—				6	2 2 1^4	0 1 0 0
D_4	(—; 8)	1	1	$\varnothing$	8	1^8	0 0 0 0

$D_5 = \mathrm{SO}(10)$

Levi	$(\nu^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*	
0	(1^5; 0)	1	1	1	1 2	9 1	2 2 2 2 2
A_1	(2 1 1 1; 0)	5	1	1	1 2	7 3	2 2 0 2 2
D_2	(1 1 1; 4)	1	1	$\varnothing$	1 4	7 1 1 1	2 2 2 0 0
A_1+A_1	(2 2 1; 0)	5	1	1	2	5 5	0 2 0 2 2
A_1+D_2	(2 1; 4)	2	1	$\varnothing$ $\}$	1 2 4	5 3 1 1	2 0 2 0 0
A_2	(3 1 1; 0)	4	2	1 3 $\}$			
	—·—·—				1 4	5 2 2 1	2 1 0 1 1
D_3	(1 1; 6)	1	1	$\varnothing$	1 6	5 1^5	2 2 0 0 0
A_1+A_2	(3 2; 0)	4	1	3	4	4 4 1 1	0 2 0 1 1
A_2+D_2	(3; 4)	1	1	$\varnothing$	3 4	3 3 3 1	0 0 2 0 0
A_3	(4 1; 0)	3	1	1	2	3 3 2 2	0 1 0 1 1
A_1+D_3	(2; 6)	1	1	$\varnothing$	2 6	3 3 1^4	0 2 0 0 0
	—·—·—				1 6	3 2 2 1^3	1 0 1 0 0
D_4	(1; 8)	1	1	$\varnothing$	1 8	3 1^7	2 0 0 0 0
A_4	(5; 0)	2	1	5	6	2^4 1 1	0 0 0 1 1
	—·—·—				8	2 2 1^6	0 1 0 0 0
D_5	(—; 10)	1	1	$\varnothing$	10	1^{10}	0 0 0 0 0

$D_6 = \mathrm{SO}(12)$

Levi	$(\nu^*; q)$	N_0	N_1	$I(\pi)$	$B(\lambda)$	λ_*	
0	(1^6; 0)	1	1	1	1 2	11,1	2 2 2 2 2 2
A_1	(2 1^4; 0)	6	1	1	1 2	9 3	2 2 2 0 2 2
D_2	(1^4; 4)	1	1	$\varnothing$	1 4	9 1 1 1	2 2 2 2 0 0
A_1+A_1	(2 2 1 1; 0)	9	1	1	1 2	7 5	2 0 2 0 2 2
A_1+D_2	(2 1 1; 4)	3	1	$\varnothing$ $\}$	1 2 4	7 3 1 1	2 2 0 2 0 0
A_2	(3 1 1 1; 0)	5	2	1 3 $\}$			
	—·—·—				1 4	7 2 2 1	2 2 1 0 1 1
D_3	(1 1 1; 6)	1	1	$\varnothing$	1 6	7 1^5	2 2 2 0 0 0
$3A_1$	(2 2 2; 0)	2×1	1	$\varnothing$	$\varnothing$	6 6	0 2 0 2 2 0
$2A_1+D_2$	(2 2; 4)	1	1	$\varnothing$ $\}$	2 4	5 5 1 1	0 2 0 2 0 0
A_1+A_2	(3 2 1; 0)	10	2	1 3 $\}$			

Table 1 (continued)

A_2+D_2	(3 1; 4)	2	1	∅	1 3 4	5 3 3 1	2 0 0 2 0 0
A_3	(4 1 1; 0)	4	1	1	1 2	5 3 2 2	2 0 1 0 1 1
A_1+D_3	(2 1; 6)	2	1	∅	1 2 6	5 3 1^4	2 0 2 0 0 0
	—·—·—				1 6	5 2 2 1^3	2 1 0 1 0 0
D_4	(1 1; 8)	1	1	∅	1 8	5 1^7	2 2 0 0 0 0
A_2+A_2	(3 3; 0)	2	1	3	3 4	4 4 3 1	0 1 1 0 1 1
A_3+A_1	(4 2; 0)	2×2	1	∅	∅	4 4 2 2	0 2 0 0 2 0
	—·—·—				6	4 4 1^4	0 2 0 1 0 0
A_3+D_2	(4; 4)	1	1	∅	4	3 3 3 3	0 0 0 2 0 0
A_2+D_3	(3; 6)	1	1	∅	3 6	3^3 1^3	0 0 2 0 0 0
A_4	(5 1; 0)	3	2	1 5	2 6	3 3 2 2 1 1	0 1 0 1 0 0
A_1+D_4	(2; 8)	1	1	∅	2 8	3 3 1^6	0 2 0 0 0 0
	—·—·—				1 6	3 2 2 2 2 1	1 0 0 0 1 1
	—·—·—				1 8	3 2 2 1^5	1 0 1 0 0 0
D_5	(1; 10)	1	1	∅	1, 10	3 1^9	2 0 0 0 0 0
A_5	(6; 0)	2×1	1	∅	∅	2^6	0 0 0 0 2 0
	—·—·—				8	2^4 1^4	0 0 0 1 0 0
	—·—·—				10	2 2 1^8	0 1 0 0 0 0
D_6	(—; 12)	1	1	∅	12	1^{12}	0 0 0 0 0 0

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Received November 21, 1977